Class of singular integral equations arising from inverse scattering problem

1977 J. Phys. A: Math. Gen. 10 7

(http://iopscience.iop.org/0305-4470/10/1/008)

J. Phys. A: Math. Gen., Vol. 10, No. 1, 1977. Printed in Great Britain. © 1977

Class of singular integral equations arising from inverse scattering problem

L Y Shih
National Research Council, Ottawa, Canada K1A OR8

Received 24 May 1976, in final form 26 August 1976

Abstract. A class of singular integral equations arising from the inverse scattering problem has been investigated. In § 2, the exact solution of the dominant equations was sought, which serves as the first approximation for the solution of the complete integral equation. In § 3, the asymptotic expansions for the phase angles appearing in the solution were derived in order to reduce the computing time for the solution. Two examples were given to illustrate the accuracy of approximations. In § 4, the method of successive approximations was used to solve the complete integral equation, and convergence of the sequence was discussed.

1. Introduction

The purpose of this paper is to investigate the solution for a class of singular integral equations of the form

$$\phi(x)+\int_{-\infty}^{\infty} \sqrt{b(x)b(s)}\, \mathrm{e}^{-\mathrm{i}\xi(x+s)}\Lambda(x, s)\phi(s)\, \mathrm{d}s = f(x) \tag{1.1}$$

where the independent variables x and s are real. In equation (1.1), the kernel is symmetric with

$$\Lambda(x, s) = \delta_+(x+s)+g(x, s),$$

where the Heisenberg delta function is defined as

$$\delta_+(x) = \frac{1}{2\pi}\lim_{\epsilon\to 0}\frac{1}{\epsilon-\mathrm{i}x},$$

and $g(x, s)$ is a complex function, regular and independent of $b(\cdot)$, with the properties:

(a) symmetric in x and s;
(b) vanishing as either $|x|$ or $|s|$ approaches infinity.

The absolute term is a complex function quadratically summable.

Equation (1.1) arises from the study of the inverse scattering problem. In an attempt to solve the Gel'fand–Levitan equation (Gel'fand and Levitan 1955), Shih (1976) introduced two families of eigenfunctions corresponding to continuous and discrete spectra, and transformed the Gel'fand–Levitan equation into a system of $N+1$ equations. It was shown that the problem of solving this system can be reduced to that of solving (1.1).

Equation (1.1) indicates that the solution of $\phi(x)$ depends upon the form of the reflection coefficients $b(x)$. They possess the basic properties as follows:

(*a*) they are Hölder continuous;

(*b*) $b(-x)=b^*(x)$ and $\phi(-x)=\phi^*(x)$ by analytic continuation, where the asterisk indicates the complex conjugate;

(*c*) $|b(x)|\leq 1$.

(*d*) they vanish at infinity more rapidly that $|x|^{-1}$.

In order to form a kernel of Cauchy type, we apply the transformation $s\to -s$ to equation (1.1). By property (*b*), the integral term may then be written as

$$\int_{-\infty}^{\infty} \sqrt{b(x)b(-s)}\, \mathrm{e}^{\mathrm{i}\xi(s-x)}\Lambda(x,-s)\phi(-s)\,\mathrm{d}s$$

$$=\int_{-\infty}^{\infty} \sqrt{b(x)b^*(s)}\, \mathrm{e}^{\mathrm{i}\xi(s-x)}\left(\frac{1}{2\pi}\lim_{\epsilon\to 0}\frac{1}{\epsilon-\mathrm{i}(x-s)}+g(x,-s)\right)\phi^*(s)\,\mathrm{d}s. \tag{1.2}$$

Separating the dominant part from this integral (Muskhelishvili 1953, p 114), equation (1.1) may be expressed as

$$\phi(x)+\frac{B(x)}{2\pi\mathrm{i}}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{\phi^*(s)}{s-x-\mathrm{i}\epsilon}\,\mathrm{d}s+\int_{-\infty}^{\infty}K(x,s)\phi^*(s)\,\mathrm{d}s=f(x), \tag{1.3}$$

where $B(x)$ denotes $|b(x)|$ for simplicity, and

$$K(x,s)=\sqrt{b(x)b^*(s)}\,\mathrm{e}^{\mathrm{i}\xi(s-x)}g(x,-s)+\mathrm{i}\frac{\sqrt{b(x)b^*(s)}\,\mathrm{e}^{\mathrm{i}\xi(s-x)}-B(x)}{2\pi(x-s)}. \tag{1.4}$$

The integral equation

$$\phi(x)+\frac{B(x)}{2\pi\mathrm{i}}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{\phi^*(s)}{s-x-\mathrm{i}\epsilon}\,\mathrm{d}s=f(x) \tag{1.5}$$

is called the dominant equation, corresponding to the integral equation (1.1).

2. Solution of the dominant equation

Define the holomorphic function $\Phi(z)$ by the Cauchy integral

$$\Phi(z)=\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\phi_0(s)}{s-z}\,\mathrm{d}s, \tag{2.1}$$

where ϕ_0 represents the solution of the dominant equation (1.5), and z is the complex variable with x as its real part. By the Plemelj formulae for the limiting value of a Cauchy integral, the dominant equation may be expressed as

$$\Phi_+(x)-\Phi_-(x)-B(x)\Phi_-^*(x)=f(x). \tag{2.2}$$

To write the real and imaginary parts of equation (2.2) separately, we have

$$(\mathrm{Re}\,\Phi)_+-(1+B)(\mathrm{Re}\,\Phi)_-=\mathrm{Re}\,f \tag{2.3}$$

and

$$(\mathrm{Im}\,\Phi)_+-(1-B)(\mathrm{Im}\,\Phi)_-=\mathrm{Im}\,f. \tag{2.4}$$

Let us introduce two holomorphic functions $U(z)$ and $V(z)$ such that

$$\frac{U_+(x)}{U_-(x)} = 1 + B(x), \qquad \frac{V_+(x)}{V_-(x)} = 1 - B(x). \tag{2.5}$$

Hence, equations (2.3) and (2.4) become

$$\left(\frac{\operatorname{Re}\Phi}{U}\right)_+ - \left(\frac{\operatorname{Re}\Phi}{U}\right)_- = \frac{\operatorname{Re} f}{U_+} \tag{2.6}$$

and

$$\left(\frac{\operatorname{Im}\Phi}{V}\right)_+ - \left(\frac{\operatorname{Im}\Phi}{V}\right)_- = \frac{\operatorname{Im} f}{V_+}. \tag{2.7}$$

The solutions of equations (2.5), called the fundamental functions, may be expressed as (Muskhelishvili 1953, pp 86–91)

$$\ln U(z) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\ln(1 + B(s))}{s - z} \, \mathrm{d}s, \tag{2.8}$$

$$\ln V(z) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\ln(1 - B(s))}{s - z} \, \mathrm{d}s. \tag{2.9}$$

By the Plemelj formulae, the limiting values of $U(z)$ and $V(z)$ may be expressed as

$$U_+(x) = \sqrt{1 + B(x)} \exp(\mathrm{i}\theta_U(x)), \tag{2.10}$$

$$V_+(x) = \sqrt{1 - B(x)} \exp(\mathrm{i}\theta_V(x)), \tag{2.11}$$

where the phase angles θ_U and θ_V are defined as the real functions:

$$\theta_U(x) = \frac{1}{2\pi} \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{\mathrm{d}\nu}{\nu} \ln\left(\frac{1 + B(x - \nu)}{1 + B(x + \nu)}\right), \tag{2.12}$$

$$\theta_V(x) = \frac{1}{2\pi} \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{\mathrm{d}\nu}{\nu} \ln\left(\frac{1 - B(x - \nu)}{1 - B(x + \nu)}\right). \tag{2.13}$$

Since $B(x)$ is an even function of x, equations (2.12) and (2.13) indicate that both $\theta_U(x)$ and $\theta_V(x)$ are odd functions of x, which implies that $\theta_U(0) = \theta_V(0) = 0$.

Now we are able to formulate the non-homogeneous Hilbert problem (2.2) in terms of the fundamental functions. Note that the contour is a straight line extending to infinity. Thus, by equations (2.6) and (2.7), the solution of the non-homogeneous Hilbert problem (2.2) may be established as (Muskhelishvili 1953, pp 92–4)

$$\Phi(z) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}s}{s - z} \left(\frac{U(z)}{U_+(s)} \operatorname{Re} f(s) + \mathrm{i} \frac{V(z)}{V_+(s)} \operatorname{Im} f(s)\right) + C(U(x) + \mathrm{i}\, V(z)) \tag{2.14}$$

where C is the constant of integration. By the Plemelj formulae and expressions (2.10) and (2.11), the limiting values of $\Phi(z)$ on the real axis can readily be obtained. This leads to the solution of dominant equation (1.5):

$$\phi_0(x)=\frac{(2-B^2(x))f(x)-B(x)f^*(x)}{2(1-B^2(x))}+\frac{B(x)}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}s}{s-x}\left(\frac{\sin(\theta_U(x)-\theta_U(s))}{[(1+B(x))(1+B(s))]^{1/2}}\mathrm{Re}\,f(s)\right.$$
$$\left.-\mathrm{i}\frac{\sin(\theta_V(x)-\theta_V(s))}{[(1-B(x))(1-B(s))]^{1/2}}\mathrm{Im}\,f(s)\right)$$
$$+CB(x)\left(\frac{\exp(\mathrm{i}\theta_U(x))}{(1+B(x))^{1/2}}-\mathrm{i}\frac{\exp(\mathrm{i}\theta_V(x))}{(1-B(x))^{1/2}}\right). \tag{2.15}$$

3. Asymptotic expansions of phase angles

The phase angles θ_U and θ_V, as given in expressions (2.12) and (2.13), depend upon the functional form of $B(x)$, and cannot in general be integrated analytically. Therefore approximate expressions for the phase angles, if obtainable, will drastically reduce the computing time for $\phi(x)$. Figures 1 and 2 show variations of the phase angles for two illustrative cases:

$$B(x)=\tfrac{1}{2}\sqrt{\pi}\exp(-x^2), \tag{3.1}$$

$$B(x)=\tfrac{1}{2}\sqrt{\pi}\tanh a[\mathrm{sech}(x+a)+\mathrm{sech}(x-a)] \qquad \text{with } a=2. \tag{3.2}$$

All of these curves exhibit monotonic decreases for large x, so that some sort of asymptotic expansions may be attainable. By the property that $B(x)$ is an even function, the integrals in expressions (2.12) and (2.13) may be expressed as

$$\lim_{\epsilon\to 0}\int_{\epsilon}^{\infty}\frac{\mathrm{d}\nu}{\nu}\ln\left(\frac{1\pm B(x-\nu)}{1\pm B(x+\nu)}\right)=2x\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{d}\nu}{x^2-\nu^2}\ln(1\pm B(\nu)), \tag{3.3}$$

where P indicates the principal value. For large x, $B(x)\ll 1$; we consider the integrand on the right-hand side of equation (3.3) is negligibly small for $\nu\gg x$, and expand

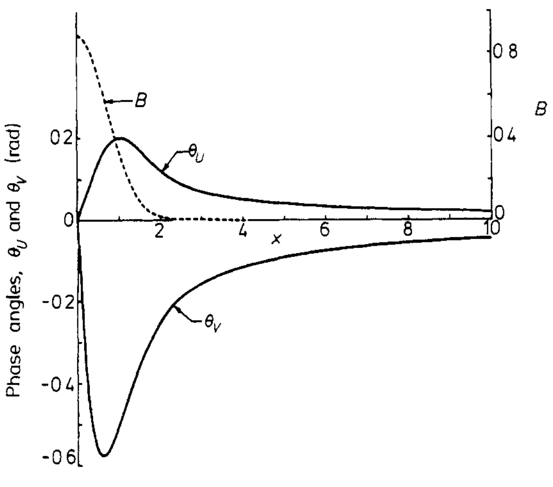

Figure 1. Phase angles for $B=\frac{1}{2}\sqrt{\pi}\exp(-x^2)$.

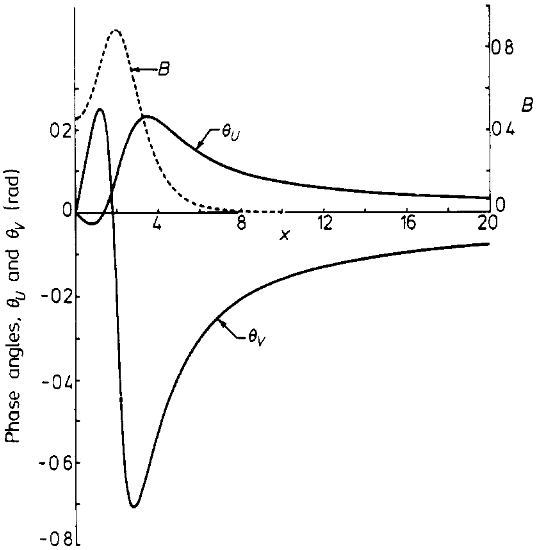

Figure 2. Phase angles for $B=\frac{1}{2}\sqrt{\pi}\tanh 2[\operatorname{sech}(x+2)+\operatorname{sech}(x-2)]$.

$(x^2-\nu^2)^{-1}$ by the binomial theorem. Thus, the phase angles may be approximated by

$$\theta_U(x)\sim\frac{1}{\pi}\sum_{n=0}^{N}x^{-(2n+1)}\int_0^\infty \nu^{2n}\ln(1+B(\nu))\,d\nu, \tag{3.4}$$

$$\theta_V(x)\sim\frac{1}{\pi}\sum_{n=0}^{N}x^{-(2n+1)}\int_0^\infty \nu^{2n}\ln(1-B(\nu))\,d\nu, \tag{3.5}$$

where N denotes the optimum degree of polynomial. The larger value of N does not necessarily yield the better result. In figures 3 and 4 the percentage errors, defined as $|\theta_{\text{approx}}/\theta_{\text{exact}}-1|\times 100$, are plotted on logarithmic scales against x. Note that in these

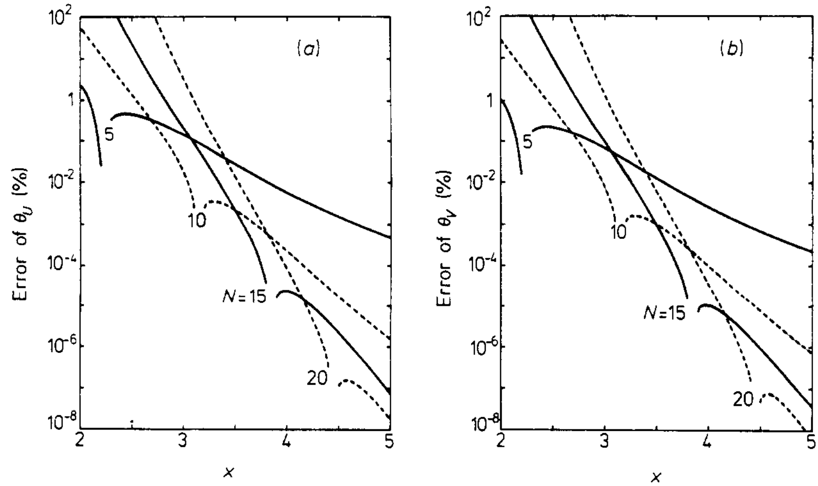

Figure 3. Percentage errors of phase angles for $B=\frac{1}{2}\sqrt{\pi}\exp(-x^2)$.

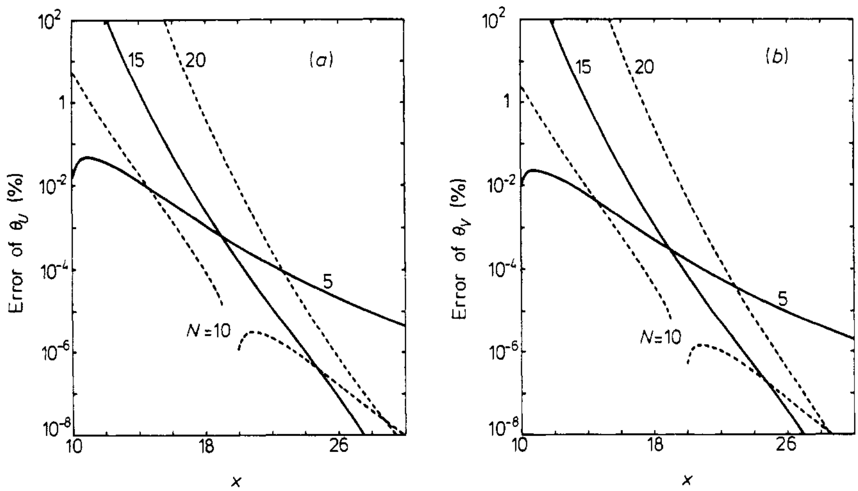

Figure 4. Percentage errors of phase angles for $B=\frac{1}{2}\sqrt{\pi}\tanh 2[\operatorname{sech}(x+2)+\operatorname{sech}(x-2)]$.

figures, discontinuity of the curves indicates change of the sign of error. Taking expression (3.2) as an example, of the four curves $N=10$ yields the best result if a fractional error of $10^{-7\cdot5}$ is allowed.

4. Solution of complete equation

The integral equation (1.1) may be solved by the method of successive approximations, starting with ϕ_0, given in equation (2.15), as the first approximation. For this purpose, we rewrite equation (1.3) in the form

$$\phi_n(x)+\frac{B(x)}{2\pi \mathrm{i}}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{\phi_n^*(s)}{s-x-\mathrm{i}\epsilon}\,\mathrm{d}s=F_{n-1}(x), \tag{4.1}$$

$$F_n(x)=f(x)-\int_{-\infty}^{\infty}K(x,s)\phi_n^*(s)\,\mathrm{d}s, \tag{4.2}$$

where $n>0$. Since equation (4.1) resembles, in form, equation (1.5), it can be solved by using expression (2.15) with f replaced by F_{n-1}. Thus, what remains to be verified is that ϕ_n must be a Cauchy sequence.

Let us define the difference

$$D_n(x)=\phi_{n+1}(x)-\phi_n(x). \tag{4.3}$$

By expression (2.15), it is easy to verify

$$\operatorname{Re} D_n(x)=-\frac{2+B(x)}{2(1+B(x))}\int_{-\infty}^{\infty}\operatorname{Re}(K^*(x,s)D_{n-1}(s))\,\mathrm{d}s$$
$$-\frac{B(x)}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}s}{s-x}\,\frac{\sin(\theta_U(x)-\theta_U(s))}{[(1+B(x))(1+B(s))]^{1/2}}\int_{-\infty}^{\infty}\operatorname{Re}(K^*(s,r)D_{n-1}(r))\,\mathrm{d}r, \tag{4.4}$$

$$\operatorname{Im} D_n(x) = \frac{2-B(x)}{2(1-B(x))} \int_{-\infty}^{\infty} \operatorname{Im}(K^*(x, s) D_{n-1}(s))\, ds$$
$$-\frac{B(x)}{2\pi} \int_{-\infty}^{\infty} \frac{ds}{s-x} \frac{\sin(\theta_V(x)-\theta_V(s))}{[(1-B(x))(1-B(s))]^{1/2}} \int_{-\infty}^{\infty} \operatorname{Im}(K^*(s, r) D_{n-1}(r))\, dr. \tag{4.5}$$

Equations (4.4) and (4.5) indicate that convergence of the successive approximations is entirely irrelevant to the form of $f(x)$.

Suppose that the function $g(x, s)$, related to $K(x, s)$ by expression (1.4), satisfies the relation

$$\int_{-\infty}^{\infty} K^*(x, s) F(s)\, ds = \eta(x) F(x), \tag{4.6}$$

where $F(x)$ denotes an arbitrary function vanishing at infinity, and $|\eta(x)|$ is less than unity for all x. Thus, by relation (4.6), equation (4.4) becomes

$$\operatorname{Re} D_n(x) = -\frac{2+B(x)}{2(1+B(x))} \operatorname{Re}(\eta(x) D_{n-1}(x))$$
$$-\frac{B(x)}{2\pi(1+B(x))^{1/2}} \int_{-\infty}^{\infty} \frac{ds}{s-x} \frac{\sin(\theta_U(x)-\theta_U(s))}{(1+B(s))^{1/2}} \operatorname{Re}(\eta(s) D_{n-1}(s)). \tag{4.7}$$

Similarly, equation (4.5) becomes

$$\operatorname{Im} D_n(x) = \frac{2-B(x)}{2(1-B(x))} \operatorname{Im}(\eta(x) D_{n-1}(x))$$
$$-\frac{B(x)}{2\pi(1-B(x))^{1/2}} \int_{-\infty}^{\infty} \frac{ds}{s-x} \frac{\sin(\theta_V(x)-\theta_V(s))}{(1-B(s))^{1/2}} \operatorname{Im}(\eta(s) D_{n-1}(s)). \tag{4.8}$$

We shall investigate from equations (4.7) and (4.8) whether it is true that

$$|\operatorname{Re} D_n(x)| < |\operatorname{Re}(\eta(x) D_{n-1}(x))| \tag{4.9}$$

and

$$|\operatorname{Im} D_n(x)| < |\operatorname{Im}(\eta(x) D_{n-1}(x))|. \tag{4.10}$$

If so, then for each $\epsilon > 0$ there exists an N such that

$$|D_n(x)| < \epsilon \qquad \text{if } n > N,$$

and thus the approximation scheme is convergent. We are going to discuss this problem separately in two different cases: one when the approximations fluctuate about the exact value, and the other when the approximations monotonically approach the exact value.

4.1. Approximations fluctuating about exact value

In this case we consider $\operatorname{Re} D_n(x) \operatorname{Re} D_{n-1}(x) < 0$ and $\operatorname{Im} D_n(x) \operatorname{Im} D_{n-1}(x) < 0$. If we combine equation (4.7) and relation (4.9), after re-arranging we obtain

$$\frac{(1+B(x))^{1/2}}{\mathrm{Re}(\eta(x)D_{n-1}(x))}\int_{-\infty}^{\infty}\frac{\sin(\theta_U(x)-\theta_U(s))}{s-x}\frac{\mathrm{Re}(\eta(s)D_{n-1}(s))}{(1+B(s))^{1/2}}\,\mathrm{d}s<\pi. \tag{4.11}$$

Similarly, by equation (4.8) and relation (4.10), we have

$$\frac{(1-B(x))^{1/2}}{\mathrm{Im}(\eta(x)D_{n-1}(x))}\int_{-\infty}^{\infty}\frac{\sin(\theta_V(x)-\theta_V(s))}{s-x}\frac{\mathrm{Im}(\eta(s)D_{n-1}(s))}{(1-B(s))^{1/2}}\,\mathrm{d}s<\pi\left(\frac{4}{B(x)}-3\right). \tag{4.12}$$

4.2. Approximations monotonically approach exact value

In this case we consider $\mathrm{Re}\,D_n(x)\,\mathrm{Re}\,D_{n-1}(x)>0$ and $\mathrm{Im}\,D_n(x)\,\mathrm{Im}\,D_{n-1}(x)>0$. If we combine equation (4.7) and relation (4.9), after re-arranging we obtain

$$\frac{(1+B(x))^{1/2}}{\mathrm{Re}(\eta(x)D_{n-1}(x))}\int_{-\infty}^{\infty}\frac{\sin(\theta_U(x)-\theta_U(s))}{x-s}\frac{\mathrm{Re}(\eta(s)D_{n-1}(s))}{(1+B(s))^{1/2}}\,\mathrm{d}s<\pi\left(3+\frac{4}{B(x)}\right). \tag{4.13}$$

Similarly, by equation (4.8) and relation (4.10), we have

$$\frac{(1-B(x))^{1/2}}{\mathrm{Im}(\eta(x)D_{n-1}(x))}\int_{-\infty}^{\infty}\frac{\sin(\theta_V(x)-\theta_V(s))}{s-x}\frac{\mathrm{Im}(\eta(s)D_{n-1}(s))}{(1-B(s))^{1/2}}\,\mathrm{d}s>\pi. \tag{4.14}$$

Relations (4.11) and (4.12) or relations (4.13) and (4.14) must be satisfied in order to reach convergence. As shown in figures 1 and 2 for two typical examples, values of the phase angles are usually small. Thus, the left-hand sides of relations (4.11) to (4.14) are normally expected to be less than the order of unity, and some may even be negative. It is obvious that the right-hand sides of relations (4.11) and (4.12) are either equal or greater than π, while the right-hand side of relation (4.13) is greater than 7π. The only one that may cause problems is relation (4.14).

5. Conclusions

A class of singular integral equation arising from the study of the inverse scattering problem has been investigated. The exact solution of the dominant equation was obtained, which serves as the first approximation for the solution of the complete integral equation. The asymptotic expansions of the phase angles appearing in the solution were also worked out in order to reduce significantly the computing time for the solution. The method of successive approximations was then proposed to solve the complete integral equation, and its convergence was discussed.

References

Gel'fand I M and Levitan B M 1955 *Am. Math. Soc. Trans., Ser. 2* **1** 253–304
Muskhelishvili N I 1953 *Singular Integral Equations* (Groningen: Noordhoff)
Shih L Y 1976 *J. Phys. A: Math. Gen.* **9** 217–24